Conditional Value-at-Risk Constrained Portfolio Optimization

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Portfolio Hedging

- **hedging** is a process of *insuring* portfolio value against (un)predictable (probably serious) losses
- portfolio can be hedged by several strategies, e.g.:
  1. investment diversification
  2. active buying/selling assets along holding period to achieve a securing result
  3. buying (or replicating) a securing derivative

- as a result of a hedging process we obtain **adjusted portfolio payoff** with reweighted probabilities of losses
one of the possible ways of hedging is to constrain portfolio value distribution by a risk measure

risk measure $\rho$ is a functional which assigns a real number (the risk exposure) to each portfolio represented by its random yield (value) $X$:

$$\rho : \mathcal{X} \rightarrow \mathbb{R}$$

where $\mathcal{X}$ is a set of all possible portfolios represented by their yields and amount $\rho(X)$ is a risk exposure of portfolio $X$

a family of risk measures functionals exhibits a fancy properties motivated by economical background
Main-Stream Risk Measures I

- **Value-at-Risk (VaR):**
  - defined as \((1 - \alpha)\)-quantile of a portfolio loss \((-X)\)
  - answers the Q: *What is the loss which is not to be exceeded with probability of \(1 - \alpha\), e.g. 99%?*

  *Figure 1 - VaR for EUR/USD risk exposure*

  - **VaR critism:** not concerning the on-tail-shape
  - not a so called "coherent" RM (generally subaditivitity property fails)
Conditional Value-at-Risk (CVaR) - "upgraded" ver. of VaR
defined as a mean extreme loss (greater than VaR)
good properties (a coherent measure)
consider a market (economy) with $N$ risky and 1 risk-free assets

make assumptions about their price making processes:

$$dB(t) = B(t)r(t)\,dt$$
$$dS_j(t) = S_j(t)\left[\mu_j(t)\,dt + \sigma_j(t)\,dw(t)\right] \quad j \in 1, \ldots, N$$

assumption of a complete market implies existence of a unique state-price density process:

$$d\xi(t) = -\xi(t)\left[r(t)\,dt + \kappa(t)^\top\,dw(t)\right],$$

where market parameters $r(t), \mu_j(t), \sigma_j(t), \kappa(t)$ are subject to calibration process
Problem Statement

- denote $W_T$ a r.v. of $T$-time portfolio value
- maximize expected utility at investment horizon $T$
- while controlling risky exposure by CVaR measure (a kind of partial hedging):

$$
\max \mathbb{E} [u(W_T)] \\
\mathbb{E} [\xi_T W_T] \leq \xi_0 W_0 \\
CVaR_\alpha(W_T) \leq \delta W_0
$$

where

$$
CVaR_\alpha(W_T) \equiv \mathbb{E} [W_0 - W_T | W_0 - W_T \geq \text{VaR}_\alpha(W_T)] \\
\text{VaR}_\alpha(W_T) \equiv - \inf \{ c \in \mathbb{R} : \mathbb{P} (W_0 - W_T \leq c) \geq \alpha \}$$
The Equivalent Problem

- previous problem formulation is hard to solve
- due to the complex CVaR representation
- as CVaR is substituted by a much simpler functional:

\[
\begin{align*}
\max_{W_T, c} & \quad \mathbb{E} [u(W_T)] \\
\mathbb{E} [\xi_T W_T] & \leq W_0 \\
c + \frac{1}{\alpha} \mathbb{E} \left[ (W_0 - W_T - c)^+ \right] & \leq \delta W_0 \\
c & \in \mathbb{R}
\end{align*}
\]

- the problem statement transforms into two-step optimization
- but a kind of "tricky" CVaR representation disappears :)

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in optimization problems CVaR can be represented by a convex function (with respect to $c$) as stated in the next theorems:

$$G_\alpha(x, c) = c + \frac{1}{\alpha} \int_{y \in \mathbb{R}^n} (l(x, y) - c)^+ f(y) \, dy,$$

where $\alpha \in (0, 1)$ is exogenous parameter, $c \in \mathbb{R}$, $l(x, y) \in \mathbb{R}$ is a function of deterministic vector $x \in X \subseteq \mathbb{R}^n$ and random vector $y \in \mathbb{R}^m$ s with distribution $f(y)$. 
Equivalent CVaR Representation II

Theorem (Main Message)

Minimizing the $\text{CVaR}_\alpha(x)$ with respect to $x \in X$ is equivalent to minimizing the function $G_\alpha(x, c)$ with respect to $(x, c) \in X \times \mathbb{R}$ in the sense that:

$$\min_{x \in X} \text{CVaR}_\alpha(x) = \min_{(x, c) \in X \times \mathbb{R}} G_\alpha(x, c),$$

this is motivation for next proposition:
The two optimization problems:

$$\max_{x \in X} \mathbb{E}[u(x)], \text{CVaR}_\alpha(x) \leq \omega, X \in \{x \in L^2(\mathbb{P}) \mid \mathbb{E}[\xi x] \leq \phi\}$$

$$\max_{(x, c) \in X \times \mathbb{R}} \mathbb{E}[u(x)], G_\alpha(x, c) \leq \omega, X \in \{x \in L^2(\mathbb{P}) \mid \mathbb{E}[\xi x] \leq \phi\}$$

are equivalent in that sense that their objective functions achieve the same minimum values.
Optimal Terminal Wealth I

- solution as a result of a **two-stage** optimization procedure:
  1. the **first-stage** solution (with respect to variable $W_T(c)$) solved using Kuhn-Tucker framework
  2. the final solution is solution of the **second-stage** optimization (with respect to variable $c$):

$$\max_{c \in \mathbb{R}} \mathbb{E} \left[ u \left( \hat{W}_T(c) \right) \right]$$

- where $\hat{W}_T(c)$ $T$-time policy is defined as follows:
Theorem (First-stage $T$-time solution)

For each $c \in \mathbb{R}$ the optimal $T$-time wealth is defined:

$$\hat{W}_T(c) = \begin{cases} 
I(y_1 \xi_T) & \text{if } \xi_T < \xi \\
W_0 - c & \text{if } \xi \leq \xi_T < \bar{\xi} \\
I(y_1 \xi_T - \frac{y_2}{\alpha}) & \text{if } \bar{\xi} \leq \xi_T,
\end{cases}$$

where $\xi = u'(W_0 - c)/y_1$, $\bar{\xi} = (u'(W_0 - c) + \frac{y_2}{\alpha})/y_1$ and $y_1, y_2 \geq 0$ are solutions of equations array:

$$
\mathbb{E} \left[ \xi_T \hat{W}_T(y_1, y_2) \right] = W(0) \\
c + \frac{1}{\alpha} \mathbb{E} \left[ (W_0 - \hat{W}_T(y_1, y_2) - c)^+ \right] = \delta W_0 \text{ or } y_2 = 0.
$$
we observe three market-state intervals on which manager exhibits different behavior:
Dynamic Hedging Strategy I

- denote $\hat{W}(t)$ a dynamic investment strategy, which leads to the evaluated optimal $T$-time wealth.
- consider CRRA family utility function and constant market parameters (interest rate $r$ and market price of risk $\kappa$)

$$u(x) = \begin{cases} 
  x^p/p & p < 1, p \neq 0 \\
  \ln(x) & p = 0 
\end{cases}$$

- using martingale risk-neutral pricing framework
- no close-form solution, need of numerical computations
as $T$-time distribution of state-variable follows:

$$\ln \xi_T | \mathcal{F}_t \sim \mathcal{N} \left( \ln \xi_t - \left(r + \frac{1}{2} \| \kappa \|^2 \right)(T - t), \| \kappa \|^2 (T - t) \right)$$

and $t$-time optimal wealth is stated as:

$$\hat{W}(t) = \mathbb{E} \left[ \frac{\xi_T}{\xi_t} \hat{W}_T(\cdot) | \mathcal{F}_t \right],$$

where $\hat{W}_T(\cdot)$ denotes Optimal Terminal Wealth solution.
Dynamic Hedging Strategy III

Theorem ($t$-time solution)

\[ \hat{W}(t) = G1 + G2 + G3, \]

where

\[ G1 \equiv \frac{y_1^{p-1}}{\xi_t} \exp \left\{ \frac{p}{p-1} \left( \ln \xi_t + \left( \frac{\|\kappa\|^2}{2p-2} - r \right) (T - t) \right) \right\} \Phi(d1) \]

\[ G2 \equiv \frac{W(0) - c}{\xi_t} \exp \left\{ \ln \xi_t - r(T - t) \right\} \left( \Phi(d2) - \Phi(d3) \right) \]

\[ G3 \equiv \frac{1}{\xi_t} \int_{\xi}^{\infty} \left( y_1 \xi_T - \frac{y_2}{2} \right)^{1-p} d\mathbb{P}(\xi_T) \]
Dynamic Hedging Strategy IV

Theorem (cont’d)

...and

\[ d1 \equiv \frac{\ln \xi - \ln \xi_t + \left( r - \frac{1}{2} \frac{p+1}{p-1} \|\kappa\|^2 \right) (T - t)}{\|\kappa\| \sqrt{T - t}} \]

\[ d2 \equiv \frac{\ln \xi - \ln \xi_t + \left( r - \frac{1}{2} \|\kappa\|^2 \right) (T - t)}{\|\kappa\| \sqrt{T - t}} \]

\[ d3 \equiv \frac{\ln \xi - \ln \xi_t + \left( r - \frac{1}{2} \|\kappa\|^2 \right) (T - t)}{\|\kappa\| \sqrt{T - t}} \]

- t-payoff profile convergence to T-time payoff
Dynamic Hedging Strategy V

CHART: $\hat{W}(t)$ as a function of $\xi(t)$
LEGEND: $q^B(t)$, $W(0.5)$, $W(0.75)$, $W(0.95)$

CHART: $\hat{W}(T)$ vs $\hat{W}^B(T)$ as a function of $\xi(t)$
LEGEND: $W^B(T = 1)$, $W(T = 1)$
Dynamic Investment Strategy I

- denote $\hat{\theta}(t)$ the investment strategy (in terms of exposure to risky assets), which leads to the optimal wealth management

\[
\hat{\theta}(t) = - \left( \frac{(\sigma_T)^{-1} \kappa_T}{\hat{W}(t)} \right) \frac{\partial \hat{W}(t)}{\partial \xi(t)} \xi(t)
\]

\[
= - \frac{1 - \rho}{\hat{W}(t)} \hat{\theta}_B(t) \frac{\partial \hat{W}(t)}{\partial \xi(t)} \xi(t)
\]

- where $\hat{\theta}_B(t)$ denotes the Benchmark investor strategy as

\[
\hat{\theta}_B(t) = \frac{1}{1 - \rho} \left( \sigma_T \right)^{-1} \kappa_T
\]
thus we can define process $q(t)$ as the exposure to risky assets relative to benchmark

$$q(t) = \frac{\hat{\theta}(t)}{\theta^B(t)}$$

and analyze it...
Dynamic Strategy

CHART: $q(t)$ as a function of $\xi(t)$
LEGEND: $q^B(t)$, $q(0.55)$, $q(0.75)$, $q(0.95)$

- recall that process $q(t)$ express exposition to risky assets
- in **good** states (left part of chart) investor behaves similar like benchmark agent
- in **intermediate** states (left-middle part) investor closes his positions in risky assets
- in **the worse** states (right part) investor exploits a leverage effect to raise his portfolio value
we made a comparison simulation with well known Value-at-Risk based strategy developed by Basak & Shapiro (2001)

we investigated terminal payoff shape $\hat{W}(T)$ as well as risky assets exposure dynamics $q(t)$

the results show that CVaR strategy could reasonably over-perform VaR strategy in the worst market states
Comparison with other strategies II

CHART: $q(t)$ as a function of $\xi(t)$
LEGEND: VaR-RM, CVaR-RM

CHART: $\hat{W}(T)$ as a function of $\xi(t)$
LEGEND: VaR-RM, CVaR-RM
we have **defined** a CVaR portfolio optimization problem
we have **proposed** an equivalent problem definition
we have **developed** a new hedging strategy

**Outlook**
- further **investigation** of strategy props
Financial RM Fancy Properties I

- A financial RM should exhibit special properties.
- These are motivated by economical interpretation.
- In the next statement, considering portfolios \( X, Y \in \mathcal{X} \) and a constant \( \lambda \in \mathbb{R} \):
  - **Monotonicity:**
    \[ X \succeq Y \implies \rho(X) \leq \rho(Y) \]
  - **Cash-Invariance:**
    \[ \rho(X + \lambda) = \rho(X) - \lambda \]
  - **Convexity:**
    \[ \forall \lambda \in [0, 1]: \]
    \[ \rho(\lambda X + (1 - \lambda)Y) \leq \lambda \rho(X) + (1 - \lambda)Y \]
Financial RM Fancy Properties II

- **Positive Homogenity:**
  \[ \forall \lambda \geq 0 : \quad \rho(\lambda X) = \lambda \rho(X) \]

- **Subaditivity:**
  \[ \rho(X + Y) \leq \rho(X) + \rho(Y) \]